

Surfaces of Revolution with Constant Gaussian Curvature in Four-Space

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Abstract

In this paper, we show that the constant property of the Gaussian curvature of surfaces of revolution in both \mathbb{R}^4 and \mathbb{R}_1^4 depend only on the radius of rotation. We then give necessary and sufficient conditions for the Gaussian curvature of the general rotational surfaces whose meridians lie in two dimensional planes in \mathbb{R}^4 to be constant, and define the parametrization of the meridians when both the Gaussian curvature is constant and the rates of rotation are equal.

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1 Introduction

It is well known that a regular surface in \mathbb{R}^3 is zero Gaussian curvature if and only if it is a part of a developable surface. A regular surface with constant Gaussian curvature, K , is locally isometric to H^2 provided $K = -1$, \mathbb{R}^2 provided $K = 0$, and S^2 provided $K = 1$. For the surfaces of revolution in \mathbb{R}^3 , it is easy to define the parametrization of the surfaces with constant Gaussian curvature.

In recent years some mathematicians have taken an interest in the surfaces of revolution in \mathbb{R}^4 , for example V. Milosheva ([6]), U. Dursun and N. C. Turgay ([3]), K. Arslan ([1]), In [4], V. Milosheva applied invariance theory of surfaces in the four dimensional Euclidean space to the class of general rotational surfaces whose meridians lie in two-dimensional planes in order to find all minimal super-conformal surfaces. These surfaces were further studied by U. Dursun and N. C. Turgay in [3], which found all minimal surfaces by solving the differential equation that characterizes minimal surfaces. They

then determined all pseudo-umbilical general rotational surfaces in \mathbb{R}^4 . K. Arslan et.al in [1] gave the necessary and sufficient conditions for generalized rotation surfaces to become pseudo-umbilical, they also shown that each general rotational surface is a Chen surface in \mathbb{E}^4 and gave some special classes of generalized rotational surfaces as examples.

Let M be a spacelike or timelike surface in Lorentz-Minkowski three-space \mathbb{R}_1^3 generated by a one-parameter family of circular arcs, R. López in [5] shown that if its Gaussian curvature K is a nonzero constant then M is a surface of revolution, he also described the parametrizations for M when $K = 0$. In [2], by applying the \mathbb{I}_r^\pm -Gauss maps, Cuong defined the parameterizations of minimal and totally umbilical spacelike surfaces of revolution in \mathbb{R}_1^4 .

In this paper, we introduce the notions of surfaces of revolution in \mathbb{R}^4 and in Lorentz-Minkowski \mathbb{R}_1^4 , we then give necessary and sufficient conditions for the Gaussian curvature of these surfaces to be constant. We then give a differential equation that characterizes the general rotational surfaces whose meridians lie in two dimensional planes in \mathbb{R}^4 with constant Gaussian curvature. In the case that rates of rotation are equal, we can define the parametrization of the meridians of these surfaces when its Gaussian curvature is constant.

2 Preliminaries

Let M be a semi-Riemannian surface, that is, a semi-Riemannian manifold of dimension two. For a coordinate system u, v in M the components of the metric tensor (the coefficients of the first fundamental form) are traditionally denoted by

$$E = g_{11} = \langle \partial_u, \partial_u \rangle, F = g_{12} = \langle \partial_u, \partial_v \rangle, G = g_{22} = \langle \partial_v, \partial_v \rangle.$$

Since M is two-dimensional, $T_p M$ is the only tangent plane at p . Thus the sectional curvature K becomes a real-valued function on M , called Gaussian curvature of M .

Let u, v be an orthogonal coordinate system in a semi-Riemannian surface, that means $F = \langle \partial_u, \partial_v \rangle = 0$. Then (see Proposition 4.4, pp. 81, [8])

$$K = -\frac{1}{eg} \left[\varepsilon_1 \left(\frac{g_u}{e} \right)_u + \varepsilon_2 \left(\frac{e_v}{g} \right)_v \right],$$

where $e = |E|^{1/2}$, $g = |G|^{1/2}$ and $\varepsilon_1, \varepsilon_2$ are the sign of E, G , respectively.

If M is a surface immersed in a manifold of constant curvature C , the normal bundle of M has a orthonormal frame $\{\nu_i\}_{i=1, \dots, n}$ then by using the equation of Gauss we have

$$K = C + \sum_{i=1}^n K_{\nu_i},$$

where K_{ν_i} is ν_i -curvature of M associated with ν_i . For more detail, let see [7].

Therefore, if M is a surface immersed in a manifold with constant curvature we then can define the Gaussian curvature by two ways. In this paper, we only use the formula of Gaussian curvature in term of the coefficients of the first fundamental form of M , and apply this formula to define the surfaces of revolution whose Gaussian curvature are constant.

We now introduce the notion surfaces of revolution in \mathbb{R}^4 . Let C be a curve in $\text{span}\{e_1, e_2, e_3\}$ parametrized by arc-length

$$z(u) = (f(u), g(u), \rho(u), 0), \quad u \in I, \quad (1)$$

where $\rho(u) > 0$. The orbit of C under the action of the orthogonal transformations of \mathbb{R}^4 leaving the plane Oxy ,

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos v & -\sin v \\ 0 & 0 & \sin v & \cos v \end{bmatrix}, \quad v \in \mathbb{R},$$

is a surface given by

$$[SR1] \quad X(u, v) = (f(u), g(u), \rho(u) \cos v, \rho(u) \sin v), \quad u \in I, v \in [0, 2\pi). \quad (2)$$

Then $[SR_1]$ is called *surface of revolution* in \mathbb{R}^4 . That means, $[RS_1]$ is orbit of a curve by rotating it around a plane.

We also have the another kind of surface of revolution in \mathbb{R}^4 , it is the obit of a plane curve rotated around both two planes. This surface is defined as following. Let C be a regular curve in $\text{span}\{e_1, e_3\}$ parametrized by arc-length

$$r(u) = (f(u), 0, g(u), 0), \quad u \in I,$$

and

$$B = \begin{bmatrix} \cos \alpha v & -\sin \alpha v & 0 & 0 \\ \sin \alpha v & \cos \alpha v & 0 & 0 \\ 0 & 0 & \cos \beta v & -\sin \beta v \\ 0 & 0 & \sin \beta v & \cos \beta v \end{bmatrix}, \quad v \in \mathbb{R},$$

be a subgroup of the orthogonal transformations group on \mathbb{R}^4 , where α, β are positive constants and $(f(u))^2 + (g(u))^2 \neq 0$.

The orbit of C under the action of the subgroup B is a surface in \mathbb{R}^4 given by

$$[SR_2] \quad X(u, v) = (f(u) \cos \alpha v, f(u) \sin \alpha v, g(u) \cos \beta v, g(u) \sin \beta v), \quad (3)$$

which is called *General rotational surface whose meridians lie in two-dimensional planes*. Then $r(u)$ is called meridian and α, β are called the rates of rotation.

Modifying this method, we can introduce the notion surfaces of revolution in Lorentz-Minkowski space. The Lorentz-Minkowski space \mathbb{R}_1^4 is the 4-dimensional vector space $\mathbb{R}^4 = \{(x_1, \dots, x_4) : x_i \in \mathbb{R}, i = 1, \dots, 4\}$ endowed the pseudo scalar product defined by

$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{i=1}^3 x_i y_i - x_4 y_4,$$

where $\mathbf{x} = (x_1, \dots, x_4), \mathbf{y} = (y_1, \dots, y_4) \in \mathbb{R}^4$.

Let C be a spacelike (timelike) curve in $\text{span}\{e_1, e_2, e_4\}$ parametrized by arc-length,

$$z(u) = (f(u), g(u), 0, \rho(u)), \quad \rho(u) > 0, \quad u \in I.$$

The orbit of C under the action of the orthogonal transformations of \mathbb{R}_1^4 leaving the spacelike plane Oxy ,

$$A_S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cosh v & \sinh v \\ 0 & 0 & \sinh v & \cosh v \end{bmatrix}, \quad v \in \mathbb{R},$$

is a surface given by

$$[SR_3] \quad X(u, v) = (f(u), g(u), \rho(u) \sinh v, \rho(u) \cosh v), \quad u \in I, \quad v \in \mathbb{R}. \quad (4)$$

The surface $[SR_3]$ is called the *surface of revolution of hyperbolic type* in \mathbb{R}_1^4 .

Let C be a spacelike (timelike) curve in $\text{span}\{e_1, e_3, e_4\}$ parametrized by arc-length,

$$z(u) = (\rho(u), 0, f(u), g(u)), \quad \rho(u) > 0, \quad u \in I.$$

The orbit of C under the action of the orthogonal transformations of \mathbb{R}_1^4 leaving the timelike plane Ozt ,

$$A_T = \begin{bmatrix} \cos v & -\sin v & 0 & 0 \\ \sin v & \cos v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad v \in \mathbb{R},$$

is a surface given by

$$[SR_4] \quad X(u, v) = (\rho(u) \cos v, \rho(u) \sin v, f(u), g(u)), \quad v \in \mathbb{R}. \quad (5)$$

The surface $[SR_4]$ then is called the *surface of revolution of elliptic type* in \mathbb{R}_1^4 .

3 Main Results

The following theorems show that the constant property of Gaussian curvatures of surfaces of revolution in four-space depends only on the radius of rotation.

Theorem 3.1. *The Gaussian curvature of surface $[SR_1]$ is constant if and only if*

1. $\rho(u) = C_1 e^{Cu} + C_2 e^{-Cu}$, provided $K = -C^2$,
2. $\rho(u) = C_1 \sin(Cu) + C_2 \cos(Cu)$, provided $K = C^2$,
3. $\rho = C_1 u + C_2$, provided $K = 0$,

where C_1, C_2 are constant such that for each $u \in I$, $\rho(u) > 0$.

PROOF. The coefficients of the first fundamental form of $[SR_1]$ are defined

$$E = \langle X_u, X_u \rangle = 1, \quad F = \langle X_u, X_v \rangle = 0, \quad G = \langle X_v, X_v \rangle = \rho^2.$$

Therefore, the Gaussian curvature is defined

$$K = -\frac{\rho''}{\rho}.$$

Solving the equation $K = -C^2$, $(C^2, 0)$ we have the conclusion of Theorem 3.1. □

We have the similar result for surfaces of revolution in \mathbb{R}_1^4 .

Theorem 3.2. *The Gaussian curvature of Surface $[SR_3]$ (or $[SR_4]$) is constant, $K = C$, if and only if*

1. $\rho(u) = C_1 e^{\lambda u} + C_2 e^{-\lambda u}$, when $\varepsilon C = -2$
 $\lambda^2 < 0$,
2. $\rho(u) = C_1 \sin(\lambda u) + C_2 \cos(\lambda u)$, when $\varepsilon C = \lambda^2 > 0$,
3. $\rho = C_1 u + C_2$, when $C = 0$,

where C_1, C_2 are constant such that $\rho(u) > 0, u \in I$ and ε is the sign of E .

PROOF. The coefficients of the first fundamental form of $[SR_3]$ (or $[SR_4]$) are defined

$$E = (f'(u))^2 + (g'(u))^2 - (\rho'(u))^2 = \varepsilon, \quad F = 0, \quad G = (\rho(u))^2 > 0,$$

where $\varepsilon = \pm 1$. It is similar to the proof of Theorem 3.1, we have the result of this Theorem. □

For the general rotational surfaces whose meridians lie in two-dimensional planes in \mathbb{R}^4 , the following Theorem gives us a differential equation that characterizes the constant Gaussian curvature surfaces.

Theorem 3.3. *The Gaussian curvature of $[SR_2]$ is constant if and only if*

$$f(u) = \frac{\sqrt{G}}{\alpha} \cos \phi(u), \quad g(u) = \frac{\sqrt{G}}{\beta} \sin \phi(u),$$

where $\phi(u)$ are solutions of the following equation

$$G \left(\frac{\sin^2 \phi}{\alpha^2} + \frac{\cos^2 \phi}{\beta^2} \right) (\phi')^2 + 2G \sin \phi \cos \phi \left(\frac{1}{\beta^2} - \frac{1}{\alpha^2} \right) \phi' + \frac{(G')^2}{4G} \left(\frac{\cos^2 \phi}{\alpha^2} + \frac{\sin^2 \phi}{\beta^2} \right) - 1 = 0, \quad (6)$$

and

$$G = (C_1 e^{Cu} + C_2 e^{-Cu})^2, \quad \text{if } K = -C^2, \quad (7)$$

$$G = (C_1 \sin(Cu) + C_2 \cos(Cu))^2, \quad \text{if } K = C^2, \quad (8)$$

$$G = (C_1 u + C_2)^2, \quad \text{if } K = 0, \quad (9)$$

where C, C_1, C_2 are constant such that $G \neq 0$.

PROOF. The coefficients of the first fundamental form of $[SR_2]$ are defined

$$E = (f')^2 + (g')^2 = 1, \quad F = 0, \quad G = \alpha^2 f^2 + \beta^2 g^2. \quad (10)$$

We have

$$K = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}}.$$

Solving the equation $K = -C^2(C^2, 0)$, we have G . Setting

$$\alpha f(u) = \sqrt{G} \cos(\phi(u)), \quad \beta g(u) = \sqrt{G} \sin(\phi(u))$$

and substituting for (10) we then have (6). \square

In the case two rates of rotation are equal, $\alpha = \beta$, we can solve the equation (6) and find the parametrization of the meridians.

Corollary 3.4. *If $\alpha = \beta$ then the Gaussian curvature of $[SR_2]$ is constant if and only if*

1. In the case $K = C^2$,

$$f(u) = \frac{C_1 \sin(Cu) + C_2 \cos(Cu)}{\alpha} \cos \phi(u), \quad g(u) = \frac{C_1 \sin(Cu) + C_2 \cos(Cu)}{\alpha} \sin \phi(u)$$

where

$$\phi(u) = \int \frac{\sqrt{1 - C^2 [C_1 \cos(Cu) - C_2 \sin(Cu)]^2}}{C_1 \sin(Cu) + C_2 \cos(Cu)} du,$$

C_1, C_2 are constants such that the formula under the integral sign is defined.

2. In the case $K = -C^2$,

$$f(u) = \frac{C_1 e^{Cu} + C_2 e^{-Cu}}{\alpha} \cos \phi(u), \quad g(u) = \frac{C_1 e^{Cu} + C_2 e^{-Cu}}{\alpha} \sin \phi(u),$$

where

$$\phi(u) = \int \frac{\sqrt{1 - C^2 [C_1 e^{Cu} - C_2 e^{-Cu}]^2}}{C_1 e^{Cu} + C_2 e^{-Cu}} du,$$

C_1, C_2 are constants such that the formula under the integral sign is defined.

3. In the case $K = 0$,

$$f(u) = \frac{C_1 u + C_2}{\alpha} \cos \phi(u), \quad g(u) = \frac{C_1 u + C_2}{\alpha} \sin \phi(u),$$

where

$$\phi(u) = \int \frac{\sqrt{1 - C_1^2}}{C_1 u + C_2} du,$$

C_1, C_2 are constants such that $|C_1| \leq 1$ and $C_1 u + C_2 \neq 0, \forall u \in I$.

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